3.1 Partial Differential Equations :-

Partial differential equations arise in large number of physical problems.

A partial differential equation (PDE) is an equation relating an unknown function (the dependent variable) of two or more variables with one or more of its

partial derivatives with respect to those variables.
Among the most frequently encountered PDE's are the following

① Laplace equation $\nabla^2 \phi = 0$ ⇒ Homogeneous

This very important equation occurs in

a) electromagnetic phenomena including electrostatics, dielectrics, steady currents and magnetostatics.

b) hydrodynamics

c) heat flow

d) gravitation

② Poisson's eq^n $\nabla^2 \phi = -\rho/\epsilon_0$ ⇒ Non homogeneous eq^n

③ The wave (Helmholtz) and time independent diffusion eq^n $\nabla^2 \phi \pm k^2 \phi = 0$ These equations arise in (a) elastic waves in solids including vibrating strings, bars, membranes.

(b) Acoustics

(c) electromagnetic waves.

(d) Nuclear reactors and so on. see 1.1.

Some general techniques exist for solving PDE's, the most general form of which is

$$H\phi = F \quad \ldots \ldots \quad 3.1.1$$

where $H\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}, x, y, z\right)$ and F is a known f^n, vector f^n to be determined.

$\varphi \to$ a scalar or vector f^n to be determined.

Two characteristics

i) Equations are linear (for our present purpose)

ii) Equations are second order.

<u>Techniques for sol^n</u> :-

1) separation of variables to get ODE which can be solved by standard methods.

2) Integral solutions using Green's f^n technique

3) Use of integral transforms.

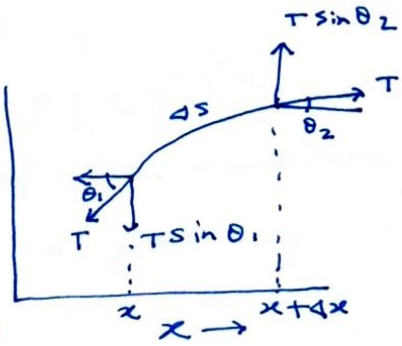
4) Numerical methods.

Since a large number of such PDE's exist we will be concerned with the solution of some typical ones.

3.2 Wave Equation :-

This is occuring in vibrating string, membranes, bars. earthquake, acoustic waves, water waves, shock waves, e.m. radiations et.c.

consider a vibrating string:
$u(x,t)$ = displacement from equilibrium
Let the tension T on the element Δs of the string be uniform.



The net upward force on Δs is

$$\Delta F = T\sin\theta_2 - T\sin\theta_1$$

If θ_1, θ_2 are small $\sin\theta_1 \sim \tan\theta_1$
$\sin\theta_2 \sim \tan\theta_2$

where the slope $\tan\theta = \frac{\partial u}{\partial x}$

Thus
$$\Delta F = T\left[\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u(x,t)}{\partial x}\right]$$
$$\cong T\left[\frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u}{\partial x^2}(x,t)\,\Delta x - \frac{\partial u}{\partial x}(x,t)\right]$$
$$= T\,\Delta x\,\frac{\partial^2 u(x,t)}{\partial x^2}$$

This upward force is by Newton's Law is the mass of the element times its upward acceleration. If ρ is mass per unit length then for small vibrations

$$\rho\Delta s \approx \rho\Delta x$$

and
$$\Delta F = \rho\Delta x\,\frac{\partial^2 u}{\partial t^2} = T\,\Delta x\,\frac{\partial^2 u}{\partial x^2}$$

Thus
$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\left(\frac{\partial^2 u}{\partial t^2}\right)$$
$$= \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{3.2.1}$$

where $c^2 = T/\rho$

The longitudinal vibration of an elastic wave is given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

with $c^2 = E/\rho$ where E = Young's modulus.
ρ = mass per unit vol.

If there is an external vertical force $f(x,t)$ per unit length in addition to tension then

$$T\frac{\partial^2 u}{\partial x^2} + f(x,t) = \rho\frac{\partial^2 u}{\partial t^2} \quad \ldots \quad (3.2.2)$$

For the vibration of a membrane

$$T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f(x,y,t) = \rho(x,y)\frac{\partial^2 u}{\partial t^2} \quad \ldots \quad 3.2.3$$

ρ is the mass per unit area of the membrane

Three dimensional wave eq^n is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$$\text{or} \quad \nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \ldots \quad (3.2.4)$$

Let us consider the general sol^n of eq 3.2.1, called the D'Alemberts soln.

Let $D_x = \frac{\partial}{\partial x}$

$D_x^2 = \frac{\partial^2}{\partial x^2}$

Then eqn 3.2.1 becomes

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} = c^2(D_x)^2 u = (cD_x)^2 u \quad \ldots \quad 3.2.5$$

Eq^n 3.2.5 may be treated as an ordinary DE with constant coeff. of the form

$$\frac{d^2 u}{dt^2} = a^2 u \quad \ldots \quad (3.2.6)$$

wher $a = cD_x$

The sol^n is (if a is a constant)

$$u = A_1 e^{at} + A_2 e^{-at} \quad \ldots \quad (3.2.7)$$

A_1 & A_2 are arbitrary constants.

Thus eq^n 3.2.6 is formally satisfied by

$$u = e^{cD_x t} F_1(x) + e^{-cD_x t} F_2(x) \quad \ldots \quad 3.2.8$$

where, since the integration has been performed w.r.t t, instead of arbitrary consts. A_1 & A_2, we have ~~ordi~~ arbitrary functions of x viz $F_1(x)$ & $F_2(x)$.

Now
$$e^{hD_x}F(x) = \left(1+hD_x+\frac{h^2}{2!}D_x^2+\cdots\right)F(x)$$
$$= F(x)+hF'(x)+\frac{h^2}{2!}F''(x)+\cdots$$
$$= F(x+h) \text{ by Taylor's expansion.}$$

Thus
$$e^{cD_xt}F_1(x) = e^{ctD_x}F_1(x) = F_1(x+ct)$$

wher $h = ct$

If $h = -ct$ then
$$e^{-cD_xt}F_2(x) = F_2(x-ct)$$

Thus the general soln is given by
$$u(x,t) = F_1(x+ct)+F_2(x-ct) \quad \ldots (3.2.9)$$

The functional form appearing in 3.2.9 shows a wave motion as the function $u(x,t)$ changes w.r.t x & t in a certain fashion.

Consider the term $F(x+ct)$

we notice if the argument is replaced by $x \Rightarrow x-c\tau$, $t \Rightarrow t+\tau$ where τ is arbitrary

$F_1(x+ct)$ remains unchanged

Hence the disturbance propagates with a velocity c to the left side i.e. in the negative x direction or backward direction.

In the same manner the function $F_2(x-ct)$ represents the disturbance to propagate in the +ve x direction with velocity c. This is forward wave.

The gen. soln is the sum of F_1 & F_2

Consider a string having one end fixed at the origin and the other end may be at a distance (large) $x=s$. consider a wave of arbitrary shape is approaching the origin given by
$$u(x,t) = F(x+ct)$$

At the origin the displacement $=0$ for all t

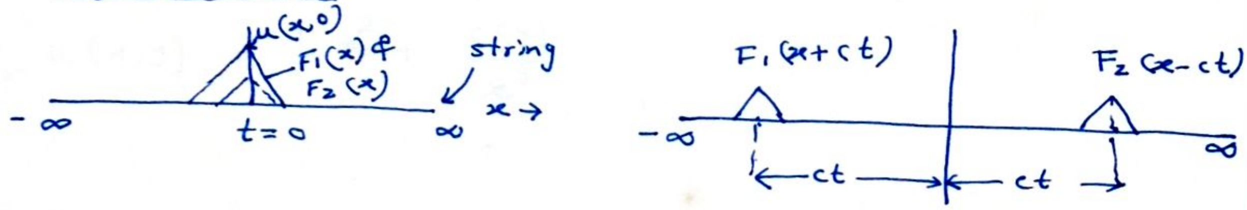
Hence the reflected wave must be of the form

$$u = -f(-x+ct)$$

since the sum of this and the original expression is zero at $x=0$ for all t. Thus transverse wave in a stretched string are inverted by reflection from a fixed end.

In physical examples usually the waves are harmonic in nature. We omit the discussion on that topic.

Fig. Infinite string



$$u(x,0) = F_1(x) + F_2(x)$$

Let at $t=0$, $u(x,0) = F_1(x) + F_2(x)$

$$= \phi(x)$$

and $t=0$. $\frac{\partial u}{\partial t} = 0$

$$\frac{\partial u}{\partial t} = \frac{\partial F_1}{\partial (x+ct)} \cdot \frac{\partial (x+ct)}{\partial t} + \frac{\partial F_2}{\partial (x-ct)} \frac{\partial (x-ct)}{\partial t}$$

$$= \frac{\partial F_1}{\partial x} \cdot c + \frac{\partial F_2}{\partial x}(-c) \quad \text{at } t=0$$

$$= 0$$

$$\therefore \frac{\partial F_1}{\partial x} = \frac{\partial F_2}{\partial x}$$

$$\Rightarrow F_1 = F_2 + c$$

$$\therefore F_1 = \frac{\phi(x)}{2} - \frac{c}{2}$$

$$F_2 = \phi(x)/2 + c/2$$

$$\Rightarrow u(x,t) = \frac{1}{2}\phi(x+ct) + \frac{1}{2}\phi(x-ct)$$

3.3 First Order PDE :

The most general firstorder PDE with two independent variables is of the form

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + c(x,y)u = R(x,y) \quad \cdots \quad 3.3.1$$

where $A(x,y)$, $B(x,y)$, $c(x,y)$ & $R(x,y)$ are given functions. If either A or $B=0$, the eq^n 3.3.1 may be solved as a first order DE discussed earlier. The arbitrary cont. of integration will be arbitrary f^ns of x or y.

Ex. consider

$$x\frac{\partial u}{\partial x} + yu = x^2 \qquad \text{where } u = u(x,y)$$

or $$\frac{\partial u}{\partial x} + \frac{yu}{x} = x$$

This is a linear eqn with ~~DE~~ I.F. is $e^{\int 3/x \, dx} = e^{3\ln x} = x^3$

Multiplying by x^3

$$x^3 \frac{\partial u}{\partial x} + 3ux^2 = x^4$$

or $\frac{\partial}{\partial x}(ux^3) = x^4$

$\therefore \; ux^3 = \int x^4 dx + c(y) = \frac{x^5}{5} + c(y)$ is the soln

$$\therefore u(x,y) = \frac{1}{5}x^2 + \frac{c(y)}{x^3}$$

3.4.

Consider the general with $C(x,y) = 0$, $R(x,y) = 0$

$$\therefore \quad A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} = 0 \quad \ldots \quad 3.4.1$$

Look for a soln $u(x,y) = f(p)$ where p is an unknown combination of x, y

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial f(p)}{\partial p} \cdot \frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f(p)}{\partial p} \cdot \frac{\partial p}{\partial y}$$

substitution in 3.4.1 gives

$$\left[A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y}\right]\frac{\partial f(p)}{\partial p} = 0$$

Then for nontrivial p [for which $\frac{\partial f(p)}{\partial p} \neq 0$]

$$A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y} = 0 \quad \ldots \quad 3.4.2$$

Let us consider the necessary condition for which $f(p)$ to remain constant as x & y vary. This means p to remain const. i.e.

$$dp = \frac{\partial p}{\partial x} \cdot dx + \frac{\partial p}{\partial y} \cdot dy = 0 \quad \ldots \quad 3.4.3$$

Eqns 3.4.2 & 3.4.3 remain identical if

$$\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)} \quad \ldots \quad 3.4.4$$

~~Interchanging~~ Integrating we can find the form of p.

Ex.

$$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$$

we seek a solⁿ of the form $u(x,y) = f(p)$

From 3.4.4 we have

$$\frac{dx}{x} = \frac{dy}{-2y}$$

or $\ln x = -\frac{1}{2}\ln y + \ln c$

or $x = c y^{-\frac{1}{2}}$

or $x y^{\frac{1}{2}} = c$

considering c as $p^{1/2}$ to avoid fractional powers we get

$$p = x^2 y$$

Thus the general solⁿ is $u(x,y) = f(x^2 y)$

Particular solⁿs can be obtained from suitable boundary conditions.

Note: In 3.4 we considered the case when $c(x,y)=0$ If a term u ~~be~~ is present so that $c(x,y) \neq 0$

In such a case we consider

$$u(x,y) = h(x,y)\, f(p) \quad \text{as the sol}^n.$$

consider the example →

$$x\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} - 2u = 0$$

Take $u(x,y) = h(x,y)\, f(p)$

$$\frac{\partial u}{\partial x} = \frac{\partial h}{\partial x} f(p) + h\frac{\partial f}{\partial x} = \frac{\partial h}{\partial x} f(p) + h\frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial h}{\partial y}\cdot f(p) + h\,\frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial y}$$

Insertion into PDE gives

$$x\frac{\partial h}{\partial x}f(p) + xh\frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial x} + z\frac{\partial h}{\partial y}f(p) + zh\frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial y} - zhf(p) = 0$$

$$\left[x\frac{\partial h}{\partial x} + z\frac{\partial h}{\partial y} - zh\right]f(p) + \left(x\frac{\partial p}{\partial x} + z\frac{\partial p}{\partial y}\right)h\frac{\partial f}{\partial p} = 0$$

The first term satisfies the PDE with u replaced by h, hence $= 0$

$$\left(x\frac{\partial p}{\partial x} + z\frac{\partial p}{\partial y}\right)h\frac{\partial f}{\partial p} = 0$$

For nontrivial $\frac{\partial f}{\partial p}$

$$\left(x\frac{\partial p}{\partial x} + z\frac{\partial p}{\partial y}\right) = 0$$

Hence $f(p)$ to remain constant as x, y vary will satisfy

$$\frac{dx}{x} = \frac{dy}{z}$$

or $\ln x = \frac{1}{2}y + \text{const.}$

or $x = ce^{y/2}$

we identify the constant with p resulting in

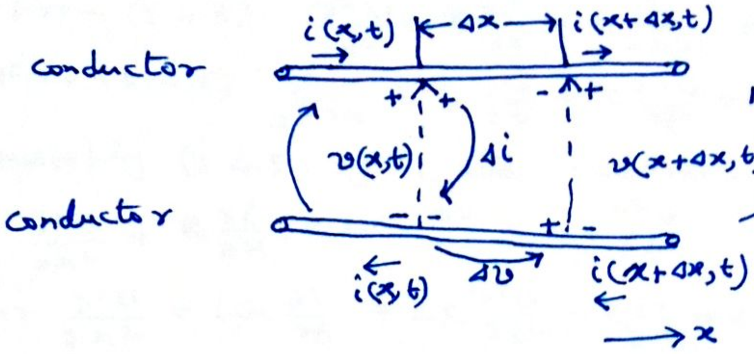
$$p = xe^{-y/2}$$

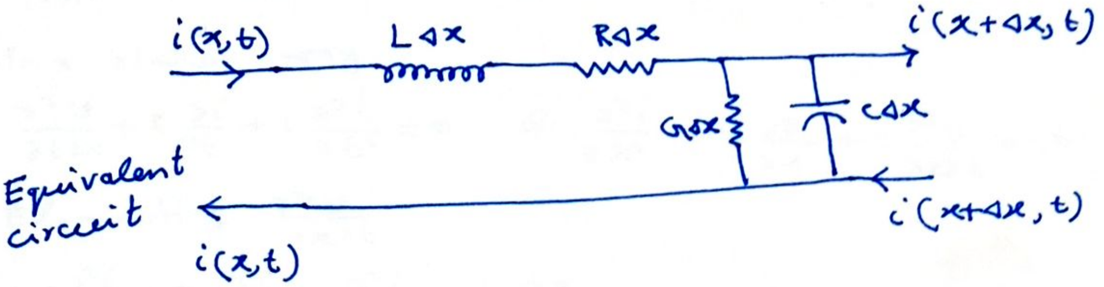
$\therefore$ Gen sol^n $u(x,y) = h(x,y)f(xe^{-y/2})$

Further Example:-

Transmission Line Equations:-

We shall consider the flow of current in a pair of linear conductors which is imperfectly insulated as shown in fig below.





Consider the current flowing as shown. Let the distance measured along the cable be x, the current and potential difference between the two wires are functions of x and t

Let $v(x,t) \to$ voltage at a point x between the line

$i(x,t) \to$ current at any point x at t

$R \to$ resistance/unit length

$L \to$ inductance/unit length

$C \to$ capacitance/unit length

$G \to$ Conductance/unit length.

Then voltage drop over Δx is given by (from KVL)

$$v(x+\Delta x,t) - v(x,t) = -iR\Delta x - L\Delta x\frac{\partial i}{\partial t}$$

$$\frac{\partial v}{\partial x}\Delta x = -iR\Delta x - L\frac{\partial i}{\partial t}\Delta x$$

$$\therefore \frac{\partial v}{\partial x} + iR + L\frac{\partial i}{\partial t} = 0 \quad \text{------} \quad (3.4.5)$$

Again the change of current over Δx is (KCL)

$$i(x+\Delta x,t) - i(x,t) = -G\Delta x v - C\Delta x\frac{\partial v}{\partial t}$$

or $$\frac{\partial i}{\partial x}\Delta x = -G\Delta x v - C\Delta x\frac{\partial v}{\partial t}$$

$$\therefore \frac{\partial i}{\partial x} + vG + C\frac{\partial v}{\partial t} = 0 \quad \text{------} \quad (3.4.6)$$

Eqns 3.4.5 & 3.4.6 are simultaneous PDE for potential difference and current in the transmission line.

From (3.4.5) $\frac{\partial^2 v}{\partial x^2} + R\frac{\partial i}{\partial x} + L\frac{\partial^2 i}{\partial x \partial t} = 0$ . . . . . . (3.4.7)

From (3.4.6) $\frac{\partial^2 v}{\partial t \partial x} + G\frac{\partial v}{\partial t} + C\frac{\partial^2 v}{\partial t^2} = 0$ . . . . . . (3.4.8)

Multiply (3.4.8) by L and subtract from (3.4.7)

$$\frac{\partial^2 v}{\partial x^2} + R\frac{\partial i}{\partial x} = LG\frac{\partial v}{\partial t} + LC\frac{\partial^2 v}{\partial t^2} \quad \text{. . . . . . . (3.4.9)}$$

or $\frac{\partial^2 v}{\partial x^2} = LG\frac{\partial v}{\partial t} + LC\frac{\partial^2 v}{\partial t^2} - R\frac{\partial i}{\partial x} = LG\frac{\partial v}{\partial t} + LC\frac{\partial^2 v}{\partial t^2} + RGv + RC\frac{\partial v}{\partial t}$

or $\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} + (LG+RC)\frac{\partial v}{\partial t} + RGv$ . . . . (3.4.10)

In a similar way

$$\frac{\partial^2 v}{\partial t \partial x} + R\frac{\partial i}{\partial t} + L\frac{\partial^2 i}{\partial t^2} = 0 \quad \& \quad \frac{\partial^2 i}{\partial x^2} + G\frac{\partial v}{\partial x} + C\frac{\partial^2 v}{\partial x \partial t} = 0$$

Eliminating $\frac{\partial^2 v}{\partial x \partial t}$

$$LC\frac{\partial^2 i}{\partial t^2} + RC\frac{\partial i}{\partial t} = \frac{\partial^2 i}{\partial x^2} + G\frac{\partial v}{\partial x}$$

or $\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + \cancel{LG}\, RC\frac{\partial i}{\partial t} + RGi + LG\frac{\partial i}{\partial t}$

$= LC\frac{\partial^2 i}{\partial t^2} + (LG+RC)\frac{\partial i}{\partial t} + RGi$ . . . . (3.4.11)

Eqns (3.4.10) and (3.4.11) are called telephone equations.
In general the equations are difficult to solve. But two special cases are of interest.

1) self inductance L and leakage due to conductance are small
i.e. $L \approx 0$, $G \simeq 0$

Then $\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t}$ . . . . (3.4.12)

$\frac{\partial^2 i}{\partial x^2} = RC\frac{\partial i}{\partial t}$ . . . . (3.4.13)

These are known as telegraph or cable equations.

2) For high frequencies the terms in the derivatives are large and some qualitative properties of solutions may be found by ignoring the effect of leakage and resistance
i.e. $G \cong 0 = R$ . . . . (3.4.14)

Thus $\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2}$ . . . . (3.4.15)

$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2}$ . . . .

These are wave equations with $1/\sqrt{LC}$ as the expression for velocity. We thus see that a transmission line with negligible resistance and leakage propagates waves of current and potential with a velocity $1/\sqrt{LC}$.
The study of eqns (3.4.5) and (3.4.6) is fundamental in the theory of electrical power transmission and telephony.
More details will be found in transmission line theory.

3.5. 2nd Order PDE :-

Most general eq^n

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = R(x,y) \quad \cdots 3.5.1$$

A, B, ... F, R, are f^ns of x, y

The equations are classified into three categorie, depending on their nature.

① If $B^2 > 4AC$ ⇒ Eq^n is hyperbolic

② $B^2 = 4AC$ ⇒ Parabolic

③ $B^2 < 4AC$ ⇒ Elliptic

If A, B, C are f^ns of (x,y) [rather than just constants] then the eq^n might be of different type in different parts of (x,y) plane. Eq^n 3.5.1 represents a large class of equations and for most class closed form of sol^n is not available. We now consider

(i) Homogeneous eq^n i.e. $R(x,y) = 0$

(ii) A, B, C, . . . F etc are constants.

Let us first consider eq^n of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = 0 \quad - - - \cdots \quad 3.5.2$$

Here the one dimensional wave eq^n

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$ and the two dimensional

Laplacian

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

are of this form but the diffusion eq^n

$$k\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$$

is not since it contains a firstorder derivative.

since eq^n 3.5.2 involves double derivative, by assuming a sol^n of the form

$u(x,y) = f(p)$ where p is some unknown function of x, y (or t) we may be able to obtain a common factor $\frac{d^2 f(p)}{dp^2}$ as the only apperarence of f in L.H.S.

Then because of the zero R.H.S all reference to the form of f may be cancelled out. We can gain some guidance on the suitable forms for the combination $p = p(x,y)$ by considering $\frac{\partial u}{\partial x}$ where u is given by $u(x,y) = f(p)$, then

$$\frac{\partial u}{\partial x} = \frac{\partial f(p)}{\partial p} \cdot \frac{\partial p}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f(p)}{\partial p} \cdot \frac{\partial p}{\partial x}\right]$$

$$= \frac{\partial f(p)}{\partial p} \cdot \frac{\partial^2 p}{\partial x^2} + \frac{\partial p}{\partial x} \frac{\partial}{\partial x}\left[\frac{\partial f(p)}{\partial p}\right]$$

$$= \frac{\partial f(p)}{\partial p} \cdot \frac{\partial^2 p}{\partial x^2} + \left(\frac{\partial p}{\partial x}\right)^2 \frac{\partial^2 f}{\partial p^2}$$

Thus double differential w.r.t x (or y) will not lead to a term on R.H.S as $\frac{\partial^2 f(p)}{\partial p^2}$ unless $\frac{\partial p}{\partial x}$ is a constant. Under this condition we must have p as a linear f^n of x or y i.e. $p = ax + by$

If we assume a sol^n

$$u(x,y) = f(ax + by)$$

then $\frac{\partial u}{\partial x} = a \frac{\partial f(p)}{\partial p}$; $\frac{\partial u}{\partial y} = b \frac{\partial f(p)}{\partial p}$

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 f(p)}{\partial p^2} \quad ; \quad \frac{\partial^2 u}{\partial x \partial y} = ab \frac{\partial^2 f(p)}{\partial p^2}$$

$$\frac{\partial^2 u}{\partial y^2} = b^2 \frac{\partial^2 f(p)}{\partial p^2}$$

substitution yields

$$(Aa^2 + Bab + Cb^2)\frac{\partial^2 f(p)}{\partial p^2} = 0 \quad \ldots \quad 3.5.3$$

This is the form we have been seeking and a solution independent of $f(p)$ can be obtained if we require a, b to satisfy

$$Aa^2 + Bab + cb^2 = 0$$

Thus

$$A\left(\frac{a^2}{b^2}\right) + B\left(\frac{a}{b}\right) + c = 0$$

$$\text{or} \quad c\left(\frac{b}{a}\right)^2 + B\left(\frac{b}{a}\right) + A = 0 \quad \ldots \ldots \quad (3.5.4)$$

The ratio $\frac{a}{b}$ should have the values

$$\frac{a}{b} = \left[-B \pm \sqrt{B^2 - 4AC}\right] / 2A$$

$$\text{or} \quad \frac{b}{a} = \left[-B \pm \sqrt{B^2 - 4AC}\right] / 2C$$

considering λ_1 and λ_2 as the two roots we get any functions of the two variables

$$p_1 = x + \lambda_1 y \; ; \quad p_2 = x + \lambda_2 y$$

as solutions of the original eqn (3.5.2). The omission of the constants in p_1 and p_2 is of no consequence as this can always be chosen in the particular form of any chosen function → only the relative weight of x & y in p is important. Since p_1 and p_2 are in general different, the general soln can be written as

$$u(x, y) = f(x + \lambda_1 y) + g(x + \lambda_2 y) \quad \ldots \ldots \quad (3.5.5)$$

where f and g are arbitrary functions.

The alternative soln given by $\frac{\partial^2 f(p)}{\partial p^2} = 0$ gives $\frac{\partial f(p)}{\partial p} = \text{const.}$

or $u(x, y) = f(p) = cp + d \sim kx + ly + m$ for which $\frac{\partial^2 f}{\partial p^2} = 0$ and all the second derivatives are individually zero.

Soln of one dimensional wave eqn

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \quad \cdots\cdots\cdots\cdots (3.5.6)$$

This is equivalent to 3.5.2 with $A = 1$, $B = 0$, $C = -\frac{1}{c^2}$

$\therefore \lambda_1$ and λ_2 are the solns of

$$1 - \frac{\lambda^2}{c^2} = 0$$

or $\lambda_1 = -c$, $\lambda_2 = c$

Thus arbitrary fns ~~are~~ of

$p_1 = x - ct$ and $p_2 = x + ct$

will be solns of (3.5.4)

$$\therefore u(x,t) = f(x-ct) + g(x+ct) \quad \cdots\cdots (3.5.5)$$

This form has been obtained for string vibration.

<u>Discussions:</u>

Let $u(x,t) = f(x-ct)$ represents the displacement of a string a time t and position x. It is clear that all positions x and time t for which $x-ct$ is a constant will have the same instantaneous displacement. But $x - ct =$ constant is exactly the relation between time and position of an observer travelling with velocity c along the +ve x direction. Consequently this moving observer sees a constant displacement of the string, whereas to a stationary observer, the initial profile $u(x,0)$ moves with a velocity c along the +ve x direction as if it was a rigid system. Thus $f(x-ct)$ represents a wave form of constant shape travelling along the +ve x axis with speed c, the actual waveform depending on function f. Similarly the term $g(x+ct)$ is a constant wave form travelling with vel c. along -ve x direction, the general soln being a superposition.

If functions f and g are same then the complete sol^n represents identical progressive waves moving in opposite directions. This may result in a wave pattern whose profile does not progress, described as a standing wave. Suppose both $f(p)$ and $g(p)$ have the form

$$f(p) = g(p) = A\cos(kp+\epsilon) \qquad k = 2\pi/\lambda \,;$$

$kc = \omega$

↳ angular freq. of the wave

eq^n 3.5.3 can be written as

$$u(x,t) = A\left[\cos\{k(x-ct)+\epsilon\} + \cos\{k(x+ct)+\epsilon\}\right]$$

$$= A\left[\cos\{(kx+\epsilon)-kct\} + \cos\{(kx+\epsilon)+kct\}\right]$$

$$= 2A\cos(kct)\cos(kx+\epsilon)$$

Here we see that the shape of the wave pattern given by x remains constant at all times but the amplitude $2A\cos(kct)$ depends on time.

At points x satisfying

$$\cos(kx+\epsilon) = 0$$

~~the~~ There is no displacement at any time representing "nodes" [at $kx+\epsilon = \pi/2,\ 3\pi/2,\ 5\pi/2$ $= (2n+1)\pi/2$, $n = 0, 1, 2, \dots$]

3.6 Three Dimensional Wave Eq^n :-

In certain co-ordinates

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \qquad \dots\dots\dots (3.6.1)$$

or $\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$

In close analogy to the one dimensional case we try the sol^n of the form

$$u(x,y,z,t) = f(p)$$

$$p = lx + my + nz + \mu t$$

It is clear that $u(x,y,z,t) = f(p)$ will be a sol^n provided

$$\left(l^2 + m^2 + n^2 - \frac{\mu^2}{c^2}\right)\frac{d^2 f(p)}{dp^2} = 0$$

As in the one dimensional case for arbitrary $f(p)$

$$l^2+m^2+n^2 = \frac{\mu^2}{c^2}$$

with a suitable normalisation we take $\mu = \pm c$ and l, m, n are such numbers that

$$l^2+m^2+n^2 = 1.$$

In other words l, m, n are the cartesian components of a unit vector $\vec{n}$ that points along the direction of propagation of the wave.

p can be written as $p = \hat{n}\cdot\vec{r} \pm ct$ and the general soln of 3.6.2 is

$$u(x,y,z,t) = f(\vec{n}\cdot\vec{r} - ct) + g(\vec{n}\cdot\vec{r} + ct) \quad \ldots (3.6.2)$$

3.7. Spherical Waves :-

Consider the wave eqn

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \quad \ldots (3.6.1)$$

we wish to consider a soln of the form

$$u = u(\underline{r}, t) \equiv u(r,t) \text{ for sph. sym. case} \quad \ldots (3.7.1)$$

where $r = |\underline{r}| = (x^2+y^2+z^2)^{1/2}$

$$r^2 = x^2+y^2+z^2$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial x} = \frac{x}{r}\frac{\partial}{\partial r}$$ for spherically symmetric case.

$$\sum_{x,y,z}\frac{\partial^2}{\partial x^2} \equiv \frac{2}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}$$ ← show this

Hence $$\nabla^2 u = \frac{2}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}$$

$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru)$$

Hence eqn 3.6.1 becomes

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \quad \ldots (3.7.2)$$

or $$\frac{\partial^2}{\partial r^2}(ru) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(ru) = 0$$

This is equivalent to the wave eqn for cartesian (one dimensional) coordinates with $u(x,t)$ replaced by $u(r,t)$.

Hence the $so\ell^n$ is

$$ru(r,t) = F_1(r+ct) + F_2(r-ct)$$

$$\text{or } u(r,t) = \frac{F_1(r+ct)}{r} + F_2(r-ct)/r \quad \cdots\cdots (3.7.3)$$

The first term on the right side of (3.7.3) represents a spherical wave converging towards the origin and second, a wave diverging from the origin with same velocity c.

3.8. General $so\ell^n$ of two dimensional Laplace eq^n.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \cdots\cdots (3.8.1)$$

We look for a $so\ell^n$ that is a f^n of $f(p)$ with $p = x + \lambda y$

From 3.5.4 we have $A = C = 1$; $B = 0$

and λ satisfies $1 + \lambda^2 = 0$

$$\therefore \lambda = \pm i$$

$$\therefore p = x \pm iy$$

Hence the general $so\ell^n$ is

$$u(x,y) = f(x+iy) + g(x-iy) \quad \cdots\cdots (3.8.2)$$

3.9 The diffusion Eq^n (one dimensional case):

Here we have

$$k\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u}{\partial t}(x,t) \quad \cdots\cdots (3.9.1)$$

$$[k] \equiv [L^2]/[t]$$

The previous method of solution viz $u(x,t) = f(p)$ fails. The general soln comes from the separation of variables.

Simple soln:

Let us set each side of 3.9.1 separately to a constant

$$k\frac{\partial^2 u}{\partial x^2} = \alpha = \frac{\partial u}{\partial t}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\alpha}{k}$$

$$\frac{\partial u}{\partial t} = \alpha$$

these have the general solution

$$u(x,t) = \frac{\alpha}{2k}x^2 + x\, g(t) + h(t)$$

& $\quad u(x,t) = \alpha t + \beta(x)$

These are compatible provided $g(t)$ is a constant or zero and $\beta(x) = \frac{\alpha}{2k}x^2 + gx$.

i.e. $u(x,t) = \frac{\alpha}{2k}x^2 + gx + \alpha t + \text{const}$.

We will treat the problem using method of separation of variables discussed just. ~~after~~.

We seek a soln of $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

such that $u \to 0$ $t \to \infty$ for all x.

Let $u(x,t) = X(x)T(t)$

$$\therefore \quad kT\frac{\partial^2 X}{\partial x^2} = X\frac{\partial T}{\partial t} \quad [\partial \equiv d]$$

Dividing by XT

$$\frac{k}{X}\frac{d^2X}{dx^2} = \frac{1}{T}\frac{dT}{dt}$$

or $\quad \frac{X''}{X} = \frac{1}{k}\frac{T'}{T}$

Let $\quad \frac{X''}{X} = \frac{1}{k}\frac{T'}{T} = -\lambda^2$

$$\therefore \quad X'' + \lambda^2 X = 0$$

$$T' + \lambda^2 kT = 0$$

Having solutions.

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

$$T(t) = C e^{-\lambda^2 kt}$$

Thus

$$u(x,t) = \{A\cos(\lambda x) + B\sin(\lambda x)\}e^{-\lambda^2 kt}$$

To satisfy the boundary condition $\lambda^2 k > 0$. Since k is real this shows $\lambda \to$ real nonzero number and the solution is sinusoidal in x.

3.10 Soln of PDE by separation of variables method :-

The idea is to reduce a more difficult problem into several simpler problems. Here we will reduce a partial DE to several ODE for which we know how to solve.

Consider the general wave eqn

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \cdots\cdots (3.10.1)$$

we consider a soln of the form

$$u(x,y,z,t) = X(x)Y(y)Z(z)T(t) \quad \cdots\cdots (3.10.2)$$

This type is said to be separable in x, y, z, t and seeking such soln is called separation of variables method.

Substitution of (3.10.2) in (3.10.1) gives

$$\frac{d^2X}{dx^2}YZT + \frac{d^2Y}{dy^2}XZT + \frac{d^2Z}{dz^2}XYT = \frac{1}{c^2}XYZ\frac{\partial^2 T}{\partial t^2}$$

In this ~~case~~ eqn three of the four functions behave as independent constant multipliers. Divide by $XYZT$ we get

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2}$$

or $$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{1}{c^2}\frac{T''}{T}$$

As X, Y, Z, T behave independently, this eqn is valid if each terms is equal to constant.

Let us put

$$\left.\begin{aligned} \frac{X''}{X} &= -l^2 \\ \frac{Y''}{Y} &= -m^2 \\ \frac{Z''}{Z} &= -n^2 \end{aligned}\right\} \quad \cdots\cdots (3.10.3)$$

$$\frac{T''}{c^2 T} = -\mu^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (3.10.3)$$

which gives $l^2 + m^2 + n^2 = \mu^2$

Eqn (3.10.3) gives four separate ODE whose solutions can be found easily in terms of the separation of constants l, m, n, μ.

The general solutions are

$$X(x) = Ae^{ilx} + Be^{-ilx}$$
$$Y(y) = ce^{imx} + De^{-imx}$$
$$Z(z) = Ee^{inx} + Fe^{-inx}$$
$$T(t) = Ge^{ic\mu t} + He^{-ic\mu t}$$

where $A, B, \ldots H$ are constants and may be determined from boundary conditions. A particular soln of the general problem is

$$u(x,y,z,t) = e^{i(lx+my+nz-c\mu t)}$$

This is a special case of the solⁿ representing a plane wave of unit amplitude having a vector $\hat{n}$ along the direction of propagation such that $\hat{n}.\vec{r} = lx + my + nz$, l, m, n are the components of the vector n in x, y, z directions.

We can write $\hat{n}$ as the wave vector $\vec{k}$ whose magnitude is $k = 2\pi/\lambda$ and $c\mu$ is the angular frequency ω
$k_x = k_y = k_z = k$
of the wave. [According to conventional theory].

Thus

$$u(x,y,z,t) \sim e^{i(k.r. \omega t)}$$

Ex.

Consider Laplace's eqⁿ in two dimension

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Let $u(x,y) = X(x)\,Y(y)$

Putting the separation constant as λ^2

$$X'' = \lambda^2 X$$
$$Y'' = -\lambda^2 Y$$

Taking $\lambda^2 > 0$ the general solution is

$$u(x,y) = \left[A\cosh(\lambda x) + B\sinh(\lambda x)\right]\left[C\cos\lambda y + D\sin\lambda y\right]$$

or $\left[Ae^{\lambda x} + Be^{-\lambda x}\right]\left[C\cos\lambda y + D\sin\lambda y\right]$ with different constants.

If $\lambda^2 < 0$ then the roles of x & y interchange. Boundary conditions will determine the constants.

3.11 Laplace Eq^n in three Dimension:

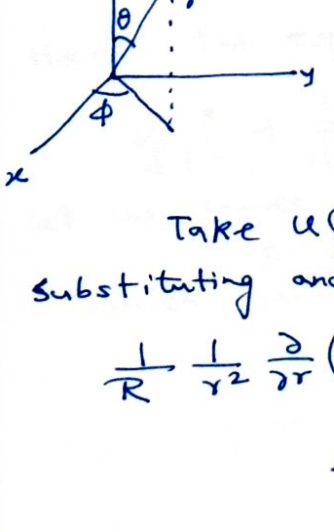
$$\nabla^2 u(x,y,z) = 0 \quad \ldots\ldots\ldots \quad 3.11.1$$

This equation has wide applicability in spherical polar co-ordinate system.

$$\nabla^2 u(x,y,z) \xrightarrow{\text{cartesian}} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\nabla^2 u(r,\theta,\phi) \xrightarrow[\text{Polar}]{\text{spherical}} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)$$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} \quad \ldots\ldots \quad 3.11.2$$



$$\nabla^2 u(r,\theta,\phi) = 0$$

Take $u(r,\theta,\phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$ $\quad\ldots\ldots\quad$ 3.11.3

substituting and dividing by $R\Theta\Phi$

$$\frac{1}{R}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta}\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)$$

$$+ \frac{1}{\Phi}\frac{1}{r^2\sin^2\theta}\frac{\partial\Phi}{\partial\phi} = 0$$

Multiply by r^2

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)$$

$$+\frac{1}{\Phi\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

The first term depends on r, the 2nd on θ and the 3rd on θ,ϕ

Hence

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)=\lambda\ (\text{say})$$

$$=-\left[\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}\right]\quad\ldots\ 3.11.4.$$

The radial eq^n is

$$2r\frac{\partial R}{\partial r}+r^2\frac{\partial^2R}{\partial r^2}-\lambda R=0$$

or $$r^2\frac{\partial^2R}{\partial r^2}+2r\frac{\partial R}{\partial r}-\lambda R=0\quad\ldots\ 3.11.5$$

Let us consider the substitution $r=\exp(t)$

then $R(r)=s(t)$

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial t}=r\frac{\partial}{\partial r}$$

$$\frac{\partial^2}{\partial t^2}=\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\cdot\frac{\partial r}{\partial t}=r\left(\frac{\partial}{\partial r}+r\frac{\partial^2}{\partial r^2}\right)$$

$$=r^2\frac{\partial^2}{\partial r^2}+r\frac{\partial}{\partial r}$$

Hence the eq^n becomes

$$\frac{d^2s}{dt^2}+\frac{ds}{dt}-\lambda s=0\quad\ldots\ 3.11.6$$

Let us make the substitution

$$s=e^{\mu t}$$

which gives $(\mu^2+\mu-\lambda)s=0$

$$\therefore\ \mu=\frac{-1\pm\sqrt{1+4\lambda}}{2}=\lambda_1\ \&\ \lambda_2$$

$\therefore$ The sol^n is

$$s = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

Here the root λ_1 & λ_2 satisfy

$$\lambda_1 + \lambda_2 = -1, \quad \lambda_1\lambda_2 = -\lambda$$

The sol^n for R becomes

$$R(r) = Ae^{\lambda_1 \ln r} + Be^{\lambda_2 \ln r}$$

$$= Ar^{\lambda_1} + Br^{\lambda_2}$$

we can choose $\lambda_1 = l$

$\lambda_2 = -l-1$

$$\therefore \lambda = l(l+1)$$

Thus $\quad R(r) = Ar^{l} + Br^{-(l+1)}$ . . . . . 3.11.7.

Note that here nothing has been assumed or proved about whether l is an integer or not.

Thus $\quad u(r,\theta,\phi) = (Ar^{l} + Br^{-l-1})\,\Theta(\theta)\,\Phi(\phi)$

with $\lambda = l(l+1)$

The eq^n for Θ & Φ is

$$\frac{1}{\Theta \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = -\lambda$$

or $\quad \dfrac{\sin\theta}{\Theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial\Theta}{\partial\theta}\right) + \lambda\sin^2\theta + \dfrac{1}{\Phi}\dfrac{\partial^2\Phi}{\partial\phi^2} = 0$

Again we separate the variables Θ & Φ and take the constant of integration as m^2.

Then $\quad \dfrac{\sin\theta}{\Theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial\Theta}{\partial\theta}\right) + l(l+1)\sin^2\theta = m^2$

and $\quad \dfrac{1}{\Phi}\dfrac{\partial^2\Phi}{\partial\phi^2} = -m^2$

The second give a soln

$$\Phi(\phi) = c'e^{im\phi} + D'e^{-im\phi}$$

$\equiv c\cos m\phi + D\sin m\phi$ 3.11.8

Single valuedness of u require that m is an integer

i.e. if $\phi \to \phi + 2\pi$ u should not change.

For $m=0$ we have

$$\frac{d^2\Phi}{d\phi^2} = 0$$

$$\therefore \Phi = C\phi + D$$

The Θ eq^n is

$$\sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \Theta}{\partial\theta}\right) + \left\{ l(l+1)\sin^2\theta - m^2 \right\} \Theta = 0$$

Let us put $\mu = \cos\theta$, $\therefore \sin\theta = (1-\mu^2)^{1/2}$

$$\therefore \frac{d\mu}{d\theta} = -\sin\theta ; \quad \frac{d}{d\theta} = \frac{d}{d\mu}\cdot\frac{d\mu}{d\theta} = -\sin\theta \frac{d}{d\mu}$$

$$= -\left[1-\mu^2\right]^{1/2} \frac{d}{d\mu}$$

Hence the Θ eq^n becomes in terms of $M(\mu)$ $[\equiv \Theta(\theta)]$

$$(1-\mu^2)^{1/2}\left[-(1-\mu^2)^{1/2}\frac{d}{d\mu}\right]\left\{-(1-\mu^2)^{1/2}(1-\mu^2)^{1/2}\frac{d}{d\mu}\right\} M(\mu)$$

$$+ \left\{ l(l+1)(1-\mu^2) - m^2 \right\} M(\mu) = 0$$

or $$(1-\mu^2)\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dM}{d\mu}\right\} + \left\{ l(l+1)(1-\mu^2) - m^2 \right\} M = 0$$

or $$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dM}{d\mu}\right\} + \left\{ l(l+1) - \frac{m^2}{(1-\mu^2)} \right\} M = 0$$

. . . . 3.11.9

This eq^n is called associated Legendre eq^n and for $m=0$, it reduces to Legendre eq^n which we have solved earlier by series method.

which gives (for $m=0$)

$$M(\mu) = E P_l(\mu) + F Q_l(\mu) \quad . \; . \; . \; . \; . \quad (3.11.10)$$

Without proof we will write the general sol^n $m \neq 0$

$$M(\mu) = E P_l^m(\mu) + F Q_l^m(\mu)$$

where $$P_l^m(\mu) = (1-\mu^2)^{|m|/2} \frac{d^{|m|}}{d\mu^{|m|}} P_l(\mu)$$

Here m must be an integer and $0 \leq m \leq l$. From analysis it is known that $Q_l^m(\mu)$ diverges at $\mu = \pm 1$ i.e. at the polar axis where $\cos\theta = \pm 1$, $\theta = 0, \pi$. We must have $F = 0$. Finite polynomial solns for Legendre eq^n requires l be an integer ≥ 0. Thus in spherical polar co-ordinate the sol^n of Laplaces eq^n is given by

$$u(r,\theta,\phi) = \left[A r^l + B r^{-(l+1)}\right]\left[C \cos m\phi + D \sin m\phi\right]$$
$$* \left[E P_l^m(\cos\theta) + F Q_l^m(\cos\theta)\right] \quad \ldots\ldots 3.11.11.$$

3.12 Sturm - Liouville theory

Eigen f^n methods for sol^n of DE

We are concerned with the sol^n of

$$L y(x) = f(x) \quad \ldots\ldots\ldots \quad 3.12.1$$

where $f(x)$ is a prescribed general f^n and suitable boundary conditions $y = y(x)$ at $x = a$ or b (say) are given. $L y(x)$ stands for a linear differential operator acting on $y(x)$.

A linear operator L has the following properties.

(i) $L(a\phi) = aL\phi$ where a is a constant.

(ii) $L(\phi_1 + \phi_2) = L\phi_1 + L\phi_2$

Ex: The derivatives $\frac{d^n}{dx^n}$ or a combination and $\int [\ ] dx$ are linear operators.

The square $(\ )^2$ and sin etc. are non linear operators

as $(ax)^2 \neq ax^2$

$\sin(\phi_1 + \phi_2) \neq \sin\phi_1 + \sin\phi_2$

A simplification results if instead of (3.12.1) we have an eqn

$$L y_i(x) = \lambda_i y_i(x) \quad \ldots\ldots \quad 3.12.2$$

for a set of functions $y_i(x)$

Eq (3.12.2) is similar in form to an eigen value eqn

$$AX^i = \lambda_i X^i \quad \ldots\ldots \quad 3.12.3.$$

where λ_i is a constant called the eigen value associated with the eigen vectors X^i, A being a linear operator

For DE's an eqn 3.12.2 is an eigen value eqn $y_i(x)$ is the eigen f^n of the differential operator L with eigen value λ_i

Ex. Consider the harmonic oscillator eqn

$$Ly = -\frac{d^2y}{dt^2} = \omega^2 y \qquad \text{where } L = -\frac{\partial^2}{\partial t^2}$$

Here $y_n = A_n e^{i\omega_n t}$

$\omega_n = 2\pi n/T$ = circular freq, T = period

Here $\omega_n^2 = n^2\omega_1^2 = n^2 (2\pi/T)^2 \Rightarrow$ eigenvalues.

sometimes an eqn of the form is obtained

$$Ly(x) = \lambda w(x) y(x) \quad \ldots \quad 3.12.4$$

where $w(x)$ is called the weight fn and should be real and does not change sign over the closed interval $a \leq x \leq b$. In most of the cases $w(x) = 1$.

We shall consider a set of functions $y(x)$ corresponding to a particular class of operators called hermitian operators, as such operators have real eigen values and occur in a large class of physical phenomena. The eigenfunctions should satisfy certain boundary conditions called Dirichlet boundary conditions.

The function should be specified at each point of the boundary.

There are other boundary conditions due to Neuman and Cauchy.

Neuman boundary conditions state that at each point of the boundary the normal derivative of the fn's should be specified.

In Cauchy boundary conditions both fn & its normal derivative should be specified.

We can consider a set of basis functions $y_n(x)$ $n = 0, 1, 2, \ldots \infty$ such that any fn $f(x)$ can be expanded within an interval in terms of these.

$$f(x) = \sum_{n=0}^{\infty} c_n y_n(x) \quad \ldots \quad (3.12.5)$$

for a different set

$$f(x) = \sum_{n=0}^{\infty} d_n z_n(x) \quad \ldots \quad (3.12.6)$$

The inner product of two f^n f & g is defined as

$$\langle f|g\rangle = \int_a^b f^*(x)\, g(x)\, dx \quad \ldots \quad (3.12.7)$$

Two f^ns are said to be orthogonal if

$$\langle f|g\rangle = 0 \quad \ldots \quad (3.12.8)$$

The norm of a f^n is given by

$$\|f\| = \langle f|f\rangle^{1/2} = \left|\int_a^b f^*(x)\, f(x)\, dx\right|^{1/2} = \left[\int_a^b |f(x)|^2 dx\right]^{1/2} \quad \ldots \quad (3.12.9)$$

An infinite dimensional vector space of functions for which an inner product is defined is called a Hilbert space. For a complete basis of independent f^ns $\phi_n(x)$, $n = 0, 1, \ldots \infty$

if

$$\langle \phi_n|\phi_m\rangle = \int_a^b \phi_n^*(x)\, \phi_m(x)\, dx = \delta_{nm} \quad n \neq m \quad \ldots (3.12.10)$$

then ϕ_n's are said to form an orthogonal set. Further if $\langle \phi_n|\phi_n\rangle = 1 \;\; \in n$

then the set is an orthonormal set.

In case we have a complete set [By a complete set we mean that the basis spaces the entire space concerned and any funtion in this space can be expanded in terms of this set] ψ_n, $n = 0, 1, \ldots \infty$. then if this set is not orthogonal, they can be made so by the so called "Gram-schmidt" orthogonalisation procedure.

We have a set of independent non orthogonal and unnormalised f^ns u_n, $n = 0, 1, 2, \ldots$

We wish to generate a set of orthogonal functions which may be normalised further.

To generate the set of independent orthogonal f^ns we do the following

for $n = 0$

let $\psi_0(x) = u_0(x)$

Normalisation yields

$$\phi_0(x) = \psi_0(x) \langle \psi_0(x) | \psi_0(x) \rangle^{-\frac{1}{2}} \quad \ldots\ldots (3.12.11)$$

For $n = 1$

let $\psi_1(x) = u_1(x) - \phi_0(x) \langle \phi_0(x) | \cancel{\phi_0} u_1(x) \rangle$

$= u_1(x) - \phi_0(x) \langle \phi_0(x) | u_1(x) \rangle$

we see that $\langle \phi_0 | \psi_1(x) \rangle = \langle \phi_0 | u_1 \rangle - \langle \phi_0 | \phi_0 \rangle \langle \phi_0 | u_1 \rangle$

Thus $\psi_1(x)$ is orthogonal to ϕ_0

we take the normalised f^n

$$\phi_1(x) = \psi_1(x) \langle \psi_1(x) | \psi_1(x) \rangle^{-\frac{1}{2}} \quad \ldots\ldots (3.12.12)$$

For $n = 2$

let $\psi_2(x) = u_2(x) - \phi_1(x) \langle \phi_1 | u_2 \rangle - \phi_0(x) \langle \phi_0 | u_2 \rangle$

We see that $\psi_2(x)$ is orthogonal to ϕ_0 and ϕ_1

$$\langle \phi_0 | \psi_2 \rangle = \langle \phi_0 | u_2 \rangle - \langle \phi_0 | \phi_1 \rangle \langle \phi_1 | u_2 \rangle - \langle \phi_0 | \phi_0 \rangle \langle \phi_0 | u_2 \rangle$$

$$\langle \phi_1 | \psi_2 \rangle = \langle \phi_1 | u_2 \rangle - \langle \phi_1 | \phi_1 \rangle \langle \phi_1 | u_2 \rangle - \langle \phi_1 | \phi_0 \rangle \langle \phi_0 | u_2 \rangle$$

$$= 0 \quad \ldots\ldots \cancel{(3.12.13)}$$

normalisation yields

$$\phi_2(x) = \psi_2(x) \langle \psi_2 | \psi_2 \rangle^{-\frac{1}{2}} \quad \ldots\ldots (3.12.13)$$

for $n=n$ we have

$$\psi_n(x) = u_n(x) - \phi_{n-1}(x)\langle \phi_{n-1}|u_n\rangle - \phi_{n-2}(x)\langle \phi_{n-2}|u_n\rangle$$

$$\dots\dots\dots \phi_0\langle \phi_0|u_n\rangle$$

This is orthogonal to $\phi_{n-1}, \phi_{n-2} \dots\dots, \phi_0$

Normalisation yields

$$\phi_n(x) = \psi_n(x)\langle \psi_n|\psi_n\rangle^{-\frac{1}{2}} \quad \dots\dots\dots \quad 3.12.14$$

This in general

$$\psi_n(x) = u_n(x) - \sum_{k=0}^{n-1} \phi_k\langle \phi_k|u_n\rangle \quad \dots\dots \quad (3.12.15)$$

Here we notice the following chart

$u_n(x)$	$\psi_n(x)$	$\phi_n(x)$
$\Downarrow$	$\Downarrow$	$\Downarrow$
linearly independent nonorthogonal unnormalised	Linearly independent orthogonal unnormalised	linearly independent orthogonal normalised (orthonormal)

Note: Gram Schmidt method is one way of making an orthonormal set. There are infinite no of other possibilities for making an orthonormal set which can be obtained by means of suitable transformation

Example

Legendre polynomial by Gram-Schmidt Procedure.

Let us form an orthonormal set from the set of functions

$$u_n = x^n \qquad n=0,1,2 \dots \quad \text{within interval } -1 \le x \le 1$$

Here $u_0 = 1$

$n=0 \quad \therefore \psi_0 = u_0 = 1$

$$\phi_0 = 1\left[\int_{-1}^{1} dx\right]^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$n=1$

$u_1 = x$

$$\psi_1 = u_1 - \phi_0\langle \phi_0|u_1\rangle$$

$$= x - \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\int_{-1}^{1} x\,dx$$

$$= x - \frac{1}{2}\left[\frac{x^2}{2}\right]_{-1}^{1} = x$$

$$\therefore \phi_1 = x\left[\int_{-1}^{1} x^2 dx\right]^{-\frac{1}{2}} = \sqrt{\frac{3}{2}}\, x$$

$$\psi_2 = u_2 - \phi_1(\phi_1|\psi_2) - \phi_0\langle\phi_0|u_2\rangle$$

$$= x^2 - \sqrt{\frac{3}{2}}\, x \cdot \sqrt{\frac{3}{2}} \int_{-1}^{1} x^3 dx - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \int_{-1}^{1} x^2 dx$$

$$= x^2 - \frac{1}{3}$$

$$\therefore \phi_2 = \left(x^2 - \frac{1}{3}\right)\left[\int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)^2 dx\right]^{-\frac{1}{2}}$$

$$\int_{-1}^{1}\left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx = \left[\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{x}{9}\right]_{-1}^{1}$$

$$= \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{2}{5} - \frac{2}{9}$$

$$= 2\left[\frac{9-5}{5.9}\right] = \frac{2.4}{5.9}$$

$$\therefore \phi_2 = \left(x^2 - \frac{1}{3}\right)\sqrt{\frac{5}{2}} \cdot \frac{3}{2} = \sqrt{\frac{5}{2}} \cdot \frac{1}{2}\left(3x^2 - 1\right)$$

and so on

This series corresponds to Legendre polynomials

$$\phi_n(x) = \sqrt{\frac{2n+1}{2}}\, P_n(x)$$

where $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$ ... and so on.

We shall list some orthogonal polynomials generated by Gram schmidt Procedure

$$u_n(x) = x^n \quad n = 0, 1, 2, \ldots\ldots$$

Polynomials	Interval	Weight Function $w(x)$	standard Normalisation
Legendre	$-1 \le x \le 1$	1	$\int_{-1}^{1}[P_n(x)]^2 dx = \frac{2}{2n+1}$
Laguerre	$0 \le x < \infty$	e^{-x}	$\int_{0}^{\infty}[L_n(x)]^2 e^{-x} dx = 1$
Hermite	$-\infty < x < \infty$	e^{-x^2}	$\int_{-\infty}^{\infty}[H_n(x)]^2 e^{-x^2} dx = 2^n \pi^{\frac{1}{2}} n!$

Any function $f(x)$ can be expanded in terms of an orthonormal basis set $\phi_n(x)$

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

To find the coefficients we take the inner product

$$\langle \phi_m(x) | f(x) \rangle = \sum_{n=0}^{\infty} c_n \left(\phi_m(x) | \phi_n(x) \right)$$

$$= \sum_{n=0}^{\infty} c_n \delta_{mn} = c_m$$

Thus $f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) = \sum_{n=0}^{\infty} \langle \phi_n(x) | f(x) \rangle \phi_n(x)$

Hermitian Operators:

Consider a linear differential operator L. The adjoint of L is defined as L^{+} and is given by

$$\int_a^b f^*(x) \left[L g(x) \right] dx = \left[\int_a^b g^*(x) \left[L^{+} f(x) \right] dx \right]^*$$

or $\langle f | L | g \rangle = \langle g | L^{+} | f \rangle^*$

$\Rightarrow \left[\langle f | L g \rangle = \langle g | L^{+} f \rangle^* \right]$

The operator is L is self adjoint or hermitian if

$L = L^{+}$

i.e. $\langle f | L | g \rangle = \langle g | L | f \rangle^*$

$$\langle g | L f \rangle^* = \langle L f | g \rangle = \langle f | L g \rangle = \langle f | L | g \rangle$$

i.e. for hermitian operators L it can act either on f or on g to keep inner product identical.

We shall now prove one important property of hermitian operators, namely their eigen values are real.

Let $Lu_i = \lambda_i u_i$

$Lu_j = \lambda_j u_j$

$\therefore \langle u_j | L | u_i \rangle = \lambda_i \langle u_j | u_i \rangle$

$\langle u_i | L | u_j \rangle = \lambda_j \langle u_i | u_j \rangle$

or $\langle u_i | L | u_j \rangle^* = \lambda_j^* \langle u_i | u_j \rangle^*$

or $\langle u_j | L | u_i \rangle = \lambda_j^* \langle u_j | u_i \rangle$

$\therefore (\lambda_i - \lambda_j^*) \langle u_j | u_i \rangle = 0$

For $j = i$ $(\lambda_i - \lambda_i^*) \langle u_i | u_i \rangle = 0$

i.e. $\lambda_i = \lambda_i^* \Rightarrow$ real eigenvalues.

3.13. Sturm Liouville Eq^n :

The general form of Sturm Liouville eq^n (for hermitian linear differential operators) is given by

$$p(x)\frac{d^2y}{dx^2} + r(x)\frac{dy}{dx} + q(x)y + \lambda f(x)y = 0 \quad \ldots \quad 3.13.1$$

where $r(x) = \frac{dp(x)}{dx}$

and p, q, r are real f^ns of x, $f(x)$ is some weight factor.

This type of eq^n can be solved by superposition methods in Green's f^n method. We can write 3.13.1 as

$$Ly = \lambda f(x) y \quad \ldots \quad 3.13.2$$

$$\text{where } L = -\left[p(x)\frac{d^2}{dx^2} + r(x)\frac{d}{dx} + q(x)\right] \quad \ldots \quad 3.13.3$$

Ex. Legendre eq^n.

$p(x) = 1 - x^2$; $r(x) = -2x \rightarrow p'(x)$, $q(x) = 0$, $f(x) = 1$

the eigenvalues are $l(l+1)$

Eq^n 3.13.1 can be rewritten as

$$(py')' + qy + \lambda \rho y = 0 \quad \text{-------------} \quad 3.13.4$$

Here $Ly = \lambda \rho y$

with $Ly = -(py')' - qy$

we shall show that under certain boundary conditions on $y(x)$, linear operators that can be written in this form are self adjoint.

Although sturm-Liouville eq^n represents only a certain class of DE's, any second order DE in the form 3.13.1 can be converted into ~~sturm~~ sturm-Liouville form by multiplying by a suitable factor.

3.14 :- Boundary Conditions :-

For linear operators of the form of sturm-Liouville eq^n to be hermitian over the range a, b requires certain boundary conditions to be satisfied.

Let y_m and y_n be two eigen f^{n}s of S.L eq^n

They must satisfy

$$\left[y_m^* p y_n' \right]_{x=a} = \left[y_m^* p y_n' \right]_{x=b} \quad \text{-------} \quad 3.14.1$$

for all m, n

Rearranging

$$\left[y_m^* p y_n' \right]_{x=b} - \left[y_m^* p y_n' \right]_{x=a} = 0 \quad \dots$$

or $$\left[y_m^* p y_n' \right]_a^b = 0 \quad \text{--------} \quad 3.14.2$$

This is an equivalent statement about the boundary conditions.

These conditions can be satisfied by say

$y(a) = y(b) = 0; \quad y(a)' = y'(b) = 0; \quad p(a) = p(b) = 0$

We shall now show that the Sturm-Liouville operator is hermitian under the boundary conditions.

We have

$$Ly = -(py')' - qy$$

using the definition for hermitian operators we have

$$I = -\int_a^b \left[y_m^* (py_n')' + y_m^* q y_n \right] dx$$

$$= -\int_a^b y_m^* (py_n')' dx - \int_a^b y_m^* q y_n \, dx$$

Integration by parts yields

$$I = -\left[y_m^* p y_n' \right]_a^b + \int_a^b (y_m^*)' p y_n' \, dx - \int_a^b y_m^* q y_n \, dx$$

The first term on R.H.S $= 0$ by boundary conditions

Integration by parts once again gives

$$I = \underbrace{(y_m^*)' p y_n \Big]_a^b}_{0} - \int_a^b \left((y_m^*)' p \right)' y_n \, dx - \int_a^b y_m^* q y_n \, dx$$

$$\therefore -\int_a^b \left[y_m^* (py_n')' + y_m^* q y_n \right] dx = -\int_a^b \left[\left((y_m^*)' p \right)' y_n + y_m^* q y_n \right] dx$$

$$= \left\{ -\int_a^b \left[y_n^* (py_m')' + y_n^* q y_m \right] dx \right\}^*$$

which proves that S.L operator L is hermitian over the prescribed interval.

3.15. Putting an Eq^n into S.L. form:

SL eq^n requires that $r(x) = p'(x)$

However any eqn of the form

$$p(x)y'' + r(x)y' + q(x)y + \lambda f(x)y = 0 \quad \ldots\ldots \quad 3.15.1$$

Can be put into self adjoint form by multiply through the I.F.

$$F(x) = \exp\left\{ \int^{x} \frac{r(z) - p'(z)}{p(z)}\, dz \right\} \quad \ldots\ldots \quad 3.15.2$$

$$\therefore\ F(x)p(x)y'' + F(x)r(x)y' + F(x)q(x)y + \lambda F(x)f(x)y = 0 \quad \ldots\ldots \quad 3.15.2$$

Now

$$(F(x)p(x)y')' = F'(x)p(x)y' + F(x)p'(x)y' + F(x)p(x)y''$$

$$F'(x)p(x)y' = F(x)\left[\frac{r(x) - p'(x)}{p(x)}\right]p(x)y'$$

$$= F(x)r(x)y' - F(x)p'(x)y'$$

Thus $(F(x)p(x)y')' = F(x)p(x)y'' + F(x)r(x)y'$

Hence from 3.15.2 we have

$$(F(x)p(x)y')' + F(x)q(x)y + \lambda F(x)f(x)y = 0 \quad \ldots 3.15.3.$$

This eq^n takes the sturm-Liouville form with a different but still non-negative weight factor.

Examples:

Hermite eq^n

$$y'' - 2xy' + 2\alpha y = 0 \quad \ldots\ldots \quad 3.15.4$$

Here $p(x) = 1$, $p'(z) = 0$, $r(z) = -2x$

$$\therefore\ IF = \exp\left[\int(-2x)dx\right] = e^{-x^2}$$

Thus the eq^n 3.15.4 becomes after multiplication by I.F

$$e^{-x^2}y'' - 2xe^{-x^2}y' + 2\alpha e^{-x^2}y = 0 \text{ or } (e^{-x^2}y')' + 2\alpha e^{-x^2}y = 0$$

which is clearly in sturm Liouville form with

$p(x) = e^{-x^2}$; $q(x) = 0$; $f(x) = e^{-x^2}$ and $\lambda = 2\alpha$

We now note several special functions which are obtained from Sturm Liouville type equations. Also we list the functions called the generating functions which generates the special functions.

1) **Legendre eqn:**

$$(1-x^2)y'' - 2xy' + l(l+1)y = [(1-x^2)y']' + l(l+1)y = 0 \quad \ldots\ldots 3.15.5$$

This is S.L type with $p(x) = 1-x^2$

$q(x) = 0$

$\rho(x) = 1$

$\lambda = l(l+1)$

The soln called Legendre polynomials $P_l(x)$ can be given by Rodrigues formula

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}(x^2-1)^l \quad \ldots\ldots 3.15.6$$

This forms an orthogonal set given by $[-1 \leq x \leq 1]$
[Normalisation can always be done]

$$\int_{-1}^{1} P_l(x) P_k(x)\,dx = \frac{2}{2l+1}\delta_{lk} \quad \ldots\ldots 3.15.7$$

The generating f^n is

$$G(x,h) = (1-2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) h^n \quad \ldots\ldots 3.15.8$$

2) **Associated Legendre eqn:**

$$[(1-x^2)y']' + \left[l(l+1) - \frac{m^2}{1-x^2}\right]y = 0 \quad \ldots\ldots 3.15.9$$

This gives Legendre eqn for $m=0$. Physically interesting solution are for $-l \leq m \leq l$ and m is an integer.
The solution regular at all finite x are called associated Legendre functions and are given by

$$P_l^m(x) = (1-x^2)^{|m|/2} \frac{d^{|m|} P_l(x)}{dx^{|m|}} \quad \ldots\ldots 3.15.10$$

$P_l^m(x) = 0$ for $m > l$

These form an orthogonal set in the range $-1 \leq x \leq 1$

$$\int_{-1}^{1} P_l^m(x)\, P_k^m(x)\, dx = \frac{2}{(2l+1)} \frac{(l+m)!}{(l-m)!} \delta_{lk} \quad \cdots \quad 3.15.11$$

Their generating function is

$$G(x,h) = \frac{(2m)!\,(1-x^2)^{m/2}}{2^m\, m!\,(1-2hx+h^2)^{m+1/2}} = \sum_{n=0}^{\infty} P_{n+m}^{m}(x)\, h^n$$

3) Bessel's Eq^n

Physical situations which yield Legendre or Associated Legendre eq^n in spherical polar co-ordinate lead to Bessel's eq^n in cylindrical co-ordinate.

Bessels' eq^n has the form

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \quad \cdots \quad 3.15.12$$

Dividing by x and changing over to $\xi = x/a^*$, the eq^n takes the SL form

$$(\xi y')' - \frac{n^2}{\xi} y + a^2 \xi y = 0 \quad \cdots \quad 3.15.13$$

[This scaling is not essential for a changeover to SL form but it gives conventional normalisation]

The prime indicates differentiation w.r.t ξ.

The solutions regular at finite x has the form

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r!\, \Gamma(n+r+1)!} \quad \cdots \quad 3.15.14$$

$\Rightarrow$ These are Bessel functions.

The orthogonality is given by

$$\int_a^b z\, J_\nu(\lambda z)\, J_\nu(kz)\, dz = 0 \quad \ldots\ldots \quad 3.15.15$$

for $\lambda \neq k$

The generating f^n is

$$G(x,h) = \exp\left[\frac{x}{2}\left(h - \frac{1}{h}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(x)\, h^n \quad \ldots\ldots \quad 3.15.16$$

4) Hermites Eq^n

This occurs for harmonic oscillator problems.

$$y'' - 2xy' + 2\alpha y = 0 \quad \ldots\ldots \quad 3.15.17$$

S.L. form

$$e^{-x^2} y'' - 2x e^{-x^2} y' + 2\alpha e^{-x^2} y = 0$$

or $$\left(e^{-x^2} y'\right)' + \left(2\alpha e^{-x^2} y\right) = 0 \quad \ldots\ldots \quad 3.15.18$$

The solns called Hermite polynomial $H_n(x)$ are given by Rodrigues formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \quad \ldots\ldots \quad 3.15.19$$

Their orthogonality over $-\alpha\, x < \infty$ is given by

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)\, H_n(x)\, dx = 2^n\, n!\, \sqrt{\pi}\, \delta_{mn} \quad \ldots \quad 3.15.20$$

The generating function is

$$G(x,h) = e^{2hx - h^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} h^n \quad \ldots\ldots \quad 3.15.21$$

5.) Laguerre Polynomial :

This eq^n appears in hydrogen atom problem and is given by

$$xy'' + (1-x)y' + ny = 0 \quad \ldots\ldots \quad 3.15.22$$

This can be converted to SL form by multiplying by the IF e^{-x}

$$x e^{-x} y'' + (1-x) e^{-x} y' + n e^{-x} y = 0$$

or $$(x e^{-x} y')' + n e^{-x} y = 0 \quad \ldots \ldots \quad 3.15.23.$$

The solutions called Laguerre polynomials are given by the Rodriguess formula

$$L_n(x) = e^{x} \frac{d^n}{dx^n} (x^n e^{-x}) \quad \ldots \ldots \quad 3.15.24$$

Their orthogonality over the range $0 \le x \le \infty$ is given by

$$\int_0^{\infty} e^{-x} L_m(x) L_n(x)\, dx = (n!)^2 \delta_{mn} \quad \ldots \ldots \quad 3.15.25$$

The generating f^n is

$$G(x,h) = \frac{e^{xh/(1-h)}}{(1-h)} = \sum_{n=0}^{\infty} \frac{L_n(x)}{n!} h^n \quad \ldots \ldots \quad 3.15.26$$